Looking at the Economics Order Quantity Model Using the Mathematics of Conic Sections

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This paper looks at the Economic Order Quantity (EOQ) model in the context of conic sections. The motivation for taking the conic sections view of EOQ is twofold:

1) The EOQ total cost curve is known to be flat, and sometimes conic section models can be parameterized in terms of their pointed-ness or eccentricity.

2) The vertex of a conic section that opens up is the minimum. All of the conic sections models have formulas for their vertices. Using the vertex formula might lead to a new derivation of the EOQ formula.

A number of classic papers have reviewed the early history of the Economic Order Quantity (EOQ) model (Erlenkotter 1990. Mennel 1961, Whitin 1954, Raymond 1931)[i]. His review did not turn up any conic sections articles. It is not immediately obvious that the conic section literature could offer any useful insights into the EOQ model. At first glance, the total cost function from the Economic Order Quantity model looks like the sum of a hyperbola and a linear function. The annual cost of operating the inventory system is given by

(1) $TC = C \bullet D + S \bullet \frac{D}{Q} + r \bullet C \bullet \frac{Q}{2}$

where the notation used is as follows:

TC =Total Cost of the Inventory System in dollars per unit time (year)

Q =Order Quantity (pieces per order)

C =Cost per unit in dollars

r = Inventory holding cost in dollars per dollar per unit time (year)

H =Annual holding costs in dollars per item per year, H = rC

A = Order cost in dollars per order

D =Demand rate pieces per unit time (year)

The simplest version of the EOQ model assumes no volume discounts so the CD term is ignored and the model is

(2) $TC = S \bullet \frac{D}{Q} + r \bullet C \bullet \frac{Q}{2}$

or

(3) $TC = S \bullet \frac{D}{Q} + H \bullet \frac{Q}{2}$

Multiplying through by Q gives

(4) $\frac{H}{2} \bullet Q^2 - TC \bullet Q + A \bullet D = 0$

a quadratic form in TC and Q with a cross product term. The cross product term indicates a rotated coordinate system. To remove the cross product term rotate the coordinate system through an angle θ. The point (Q, TC) will be rotated into the point (Q', TC') which are related as follows:

(5) $Q' = Q \bullet \cos(\theta) + TC \bullet \sin(\theta)$

$$TC' = -Q \bullet \sin(\theta) + TC \bullet \cos(\theta)$$

or inversely

(6) $Q = Q' \bullet \cos(\theta) - TC' \bullet \sin(\theta)$

$$TC = Q' \bullet \sin(\theta) + TC' \bullet \cos(\theta)$$

Substitute this second set of equations (6) into the quadratic form (4) gives the new quadratic form in terms of Q' and TC'

Coefficient of Q'^2

(7) $(\frac{H}{2})\left[\cos(\theta)^2\right] - \left|\cos(\theta)\sin(\theta)\right|$

Coefficient of $(Q')TC'$

(8) $\frac{H}{2} \bullet [-2 \bullet \cos(\theta) \bullet \sin(\theta) - \cos(\theta)^2 + \sin(\theta)^2]$

Coefficient of TC'^2

(9) $\frac{H}{2} \bullet \sin(\theta)^2 + \cos(\theta) \bullet \sin(\theta)$

If the coefficient of $(Q')TC'$ is to be zero then the angle of rotation is

(10) $\theta = \arctan(\frac{2}{H})/2$

Then the equation of the hyperbola can be given in standard form:

(11) $\frac{Q'^2}{a^2} - \frac{TC'^2}{b^2} = 1$

where

(12) $a^2 = (DS) / [(H/2) \bullet \cos(\theta)^2 - \cos(\theta) \bullet \sin(\theta)]$

(13) $b^2 = (D \bullet S) / [(H/2) \bullet (-2\cos(\theta) \bullet \sin(\theta) - \cos(\theta)^2 + \sin(\theta)^2)]$

(14) $c^2 = a^2 + b^2$

And the eccentricity of the hyperbola is given by

(15) $e = a/c$

The eccentricity of a hyperbola indicates the flatness of a hyperbola, a rectangular hyperbola

$$x \bullet y = 1$$

has an eccentricity of 1. Higher values indicate hyperbolas that are less flat (more pointed).

If one assumes that the angle of rotation θ is small, then cos(θ) is almost 1, and sin(θ) is almost zero. Then the formula for eccentricity reduces to

(16) $e = \sqrt{(1+H^2)}$

Below is a table which examines the value of the approximation. This article uses numerical examples based on an example given in Krajewski and Ritzman.

> A museum of natural history opened a gift shop two years ago. .... One of the top selling items is a birdfeeder. Sales are 18 units per week, and the supplier charges $60 per unit. The cost of placing an order with the supplier is $45. Annual holding cost is 25% of the birdfeeders value, and the museum operates 52 weeks per year....."[ii]

H	theta	theta-deg	Eccentricity	Approximate Eccentricity	Percent Error
0.01	0.783	44.9	1.56	1.00	-35.89%
0.1	0.760	43.6	1.58	1.00	-36.39%
1	0.554	31.7	1.97	1.41	-28.21%
2	0.393	22.5	2.64	2.24	-15.30%
3	0.294	16.8	3.46	3.16	-8.60%
4	0.232	13.3	4.36	4.12	-5.43%
5	0.190	10.9	5.29	5.10	-3.61%
6	0.161	9.2	6.24	6.08	-2.52%
7	0.139	8.0	7.21	7.07	-1.93%
8	0.122	7.0	8.18	8.06	-1.44%
9	0.109	6.3	9.17	9.06	-1.25%
10	0.099	5.7	10.15	10.05	-0.99%
11	0.090	5.2	11.14	11.05	-0.85%
12	0.083	4.7	12.12	12.04	-0.65%
13	0.076	4.4	13.11	13.04	-0.55%
14	0.071	4.1	14.11	14.04	-0.53%
15	0.066	3.8	15.10	15.03	-0.44%
16	0.062	3.6	16.09	16.03	-0.37%
17	0.059	3.4	17.09	17.03	-0.35%
18	0.055	3.2	18.08	18.03	-0.29%
19	0.052	3.0	19.08	19.03	-0.28%
20	0.050	2.9	20.07	20.02	-0.22%

The table suggests that the approximation is quite good for values of H above 9 or 10.

An EOQ Derivation that Almost Works

Conic sections analysis provides a formula for the vertex of a hyperbola. The vertex of a hyperbola which opens up is also the minimum. Find the vertex of the Q', TC' hyperbola in the rotated coordinate system, that is, Q'* and TC'*, and rotate that point to get to Q* and TC* in the original coordinate system. The transformed vertex is the vertex of the transformed hyperbola, but it is not the minimum. For the numerical example above, the difference between Q* and the Q resulting from transforming the vertex of the Q', TC hyperbola only .3. For large H, the rotation of the coordinate system is minimal, and the result is approximately true. For very small H, the rotation of the coordinate system can be close to –45 degrees, and the assumption is not close to being accurate.

Using the Conic Section Model for Gauging Sensitivity.

The total cost function (3) is very flat near Q*. This result is illustrated by the formula

(17) $TC/TC* = \frac{1}{2} \bullet (\frac{Q}{Q*} + \frac{Q*}{Q})$

This result is common in the EOQ literature, and the first citation is from the 1930's; however, it appears to a generalization of Menaechmus (350 B.C.) result on the rectangular hyperbola.

(18) $y/y* = x*/x$

or

(19) $x \bullet y = x* \bullet y*$

The Menaechmus result (18) or (19) can be written in transformed coordinates as follows:

(20) $y*'^2 - x*'^2 = y'^2 - x'^2$

The key to simplifying the algebra is to note that θ, the angle of rotation, is 45 degrees so that tan(θ) is 1. The same technique was attempted with equation (17). The same analysis does not give as satisfactory a result. In the Menaechmus case, the tan(θ) is known directly. In the second case, the tan(2θ) = (2/H) is known. A half angle formula

$\tan(2\theta) = 2\tan(\theta)/1 - \tan(\theta)^2$

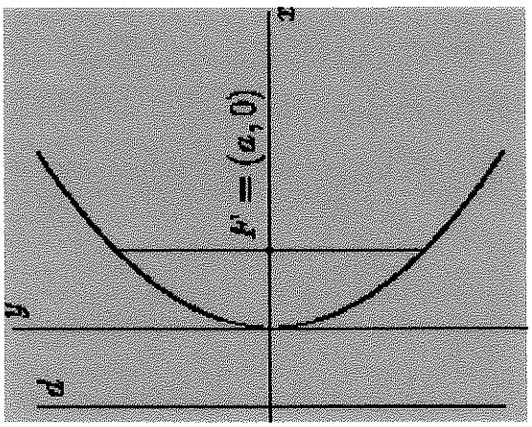
allows solution for the tan(θ), but the result is not very simple.

(21) $\frac{TC' - Q'}{TC*' - Q*'} = \frac{1}{2}\left[\frac{(Q' + \tan(\theta) \bullet TC')^2 + (Q*' + \tan(\theta) \bullet TC'*)^2}{(Q'Q*' + Q*'TC'\tan(\theta) + Q'TC*'\tan(\theta) + TC'TC*'\tan(\theta)^2)}\right]$

The flatness of the total cost formula near Q* means that users of the EOQ formula are free to experiment with Q's near Q* that are convenient, that is, are consistent with packaging, shipping, or other considerations.

The conic sections literature suggests another approach. The semi latus rectum is the distance from a focus of the hyperbola to a point on the hyperbola with the same y coordinate. The hyperbola is very flat below the focus. The formula for the semi latus rectum is

(18) $s = a^2 / b$



For the museum example, s= 4.96 ~ 5.0. The differences between TC and TC* are minimal within one semi latus rectum of Q*, less than .25%. Within 4 semi latus rectums, the difference are less than 5%. Within 8 semi latus rectums, the differences are less than 10 percent. Because, the EOQ hyperbola is tilted, departures from Q* to right have less of an impact than departures from Q* to the left.

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